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## On abstract mean ergodic theorems. II

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## ON ABSTRACT MEAN ERGODIC THEOREMS. II

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## 1. Introduction

This is a continuation of [7]. In [7] we proved abstract mean ergodic theorems for weakly right ergodic semigroups  $\mathfrak{S}$  of continuous linear operators on a *complete* locally convex topological vector space  $E$ . Among other things it was proved that the fixed points of  $\mathfrak{S}$  separate the fixed points of the adjoint semigroup  $\mathfrak{S}^* = \{T^* : T \in \mathfrak{S}\}$  if and only if  $E$  is the direct sum of the fixed points of  $\mathfrak{S}$  and the closed linear subspace of  $E$  determined by the set  $\{x - Tx : x \in E \text{ and } T \in \mathfrak{S}\}$ . This generalizes Sine's mean ergodic theorem for a single Banach space contraction operator ([5], [6], [9]). In the present paper we shall first derive a criterion, called the finite dimension criterion, for the validity of a mean ergodic theorem for a weakly right ergodic semigroup which has as a special case Sine's and Atalla's criterion for a single Banach space contraction operator ([2], [8], [11]). We shall next study ergodic properties of right ergodic semigroups of Markov operators on  $C(X)$ ,  $C(X)$  being the Banach space of all (real or complex) continuous functions on a compact Hausdorff space  $X$  with the supremum norm. Sine's results in [10] will be generalized.

## 2. Definitions and the finite dimension criterion

Throughout this section,  $E$  will be a *complete* locally convex topological vector space (t. v. s.) and  $\mathfrak{S}$  a semigroup of continuous linear operators on  $E$ . For an  $x \in E$  we denote by  $A(x)$  the affine subspace of  $E$  determined by the set  $\{Tx : T \in \mathfrak{S}\}$ , i. e.

$$A(x) = \{y : y = \sum_{i=1}^k a_i T_i x, \sum_{i=1}^k a_i = 1, T_i \in \mathfrak{S}, 1 \leq k < \infty\},$$

and by  $\bar{A}(x)$  the closure of  $A(x)$  in  $E$ . A net  $(T_n, n \in \mathcal{A})$  of linear operators on  $E$  is said to be (*weakly*) *right* [resp. (*weakly*) *left*]  *$\mathfrak{S}$ -ergodic* if it satisfies:

- (I) For every  $x \in E$  and all  $n \in \mathcal{A}$ ,  $T_n x \in \bar{A}(x)$ .
- (II) The transformations  $T_n$  are equicontinuous.
- (III) For every  $x \in E$  and all  $T \in \mathfrak{S}$ ,

$$(\text{weak-}) \lim_n T_n T x - T_n x = 0 \quad [\text{resp. } (\text{weak-}) \lim_n T T_n x - T_n x = 0].$$

The semigroup  $\mathfrak{S}$  is said to be (*weakly*) *right* [resp. (*weakly*) *left*]

*ergodic* if it possesses at least one (weakly) right [resp. (weakly) left]  $\mathfrak{S}$ -ergodic net  $(T_n, n \in \mathcal{J})$ . Whenever  $(T_n, n \in \mathcal{J})$  is a both (weakly) right and left  $\mathfrak{S}$ -ergodic net, we call it simply (weakly)  $\mathfrak{S}$ -ergodic. And if  $\mathfrak{S}$  possesses at least one (weakly)  $\mathfrak{S}$ -ergodic net,  $\mathfrak{S}$  is said to be (weakly) *ergodic*. (See [4] and [7].)

The adjoint semigroup of  $\mathfrak{S}$  is the semigroup  $\mathfrak{S}^* = \{T^* : T \in \mathfrak{S}\}$ , where  $T^*$  is the adjoint operator of  $T$  defined by  $\langle x, T^*x^* \rangle = \langle Tx, x^* \rangle$  for all  $x \in E$  and all  $x^* \in E^*$ ,  $E^*$  being the topological dual of  $E$ . We let

$$F = \{x \in E : Tx = x \text{ for all } T \in \mathfrak{S}\}$$

and

$$F^* = \{x^* \in E^* : T^*x^* = x^* \text{ for all } T^* \in \mathfrak{S}^*\}.$$

**Lemma.** *Let  $\mathfrak{S}$  be a weakly right ergodic semigroup. If  $\dim F < \infty$ , then  $\dim F^* \geq \dim F$ .*

*Proof.* Let  $f$  be any linear functional on  $F$ . Then  $f$  is continuous on  $F$ , as  $F$  is finite dimensional. Therefore by the Hahn-Banach theorem there exists an  $f^* \in E^*$  such that

$$f^* = f \text{ on } F.$$

Write

$$U = \{x \in E : |\langle x, f^* \rangle| < 1\}.$$

Now if  $(T_n, n \in \mathcal{J})$  is a weakly right  $\mathfrak{S}$ -ergodic net, then by the equicontinuity of the operators  $T_n$  we can choose a neighborhood  $W$  of the origin of  $E$  such that  $W \subset U$  and also such that

$$T_n W \subset U \text{ for all } n \in \mathcal{J}.$$

Let

$$A^* = \{x^* \in E^* : x^* = f \text{ on } F \text{ and } |\langle x, x^* \rangle| \leq 1 \text{ for all } x \in U\}$$

and

$$B^* = \{x^* \in E^* : x^* = f \text{ on } F \text{ and } |\langle x, x^* \rangle| \leq 1 \text{ for all } x \in W\}.$$

It is easily seen that  $f^* \in A^* \subset B^*$  and that

$$x^* \in A^* \text{ implies } T_n^* x^* \in B^* \text{ for all } n \in \mathcal{J}.$$

The Banach-Alaoglu theorem shows that  $B^*$  is weak\*-compact, thus there exists a subnet  $(T_{n'}^*, f^*, n' \in \mathcal{J}')$  of the net  $(T_n^* f^*, n \in \mathcal{J})$  which converges in the weak\*-topology to a point  $g^*$  in  $B^*$ . Hence for every  $T^* \in \mathfrak{S}^*$  and all  $x \in E$

$$\langle x, T^* g^* \rangle = \lim_{n'} \langle x, T^* T_{n'}^* f^* \rangle = \lim_{n'} \langle T_{n'} x, f^* \rangle$$

$$\begin{aligned}
&= \lim_{n'} \langle T_{n'} x, f^* \rangle = \lim_{n'} \langle x, T_{n'}^* f^* \rangle \\
&= \langle x, g^* \rangle.
\end{aligned}$$

It follows that  $g^* \in F^*$ , and since  $g^* = f$  on  $F$ , we immediately conclude that  $\dim F^* \geq \dim F$ . The proof is complete.

**Remark 1.** The above-given argument can easily be modified to show that if  $\mathfrak{S}$  is a weakly right ergodic semigroup, then  $\dim F^* < \infty$  implies  $\dim F \leq \dim F^*$ . Any continuous linear functional on  $F$  can be extended to a continuous linear functional on  $E$  belonging to  $F^*$ .

**Theorem 1.** Let  $\mathfrak{S}$  be a weakly right ergodic semigroup of continuous linear operators on a complete locally convex t. v. s.  $E$ . If either  $\dim F < \infty$  or  $\dim F^* < \infty$  then the following conditions are equivalent:

- (a)  $\dim F = \dim F^*$ .
- (b)  $E$  is the direct sum of  $F$  and  $N$ , where  $N$  is the closed linear subspace of  $E$  determined by the set  $\{x - Tx : x \in E \text{ and } T \in \mathfrak{S}\}$ .

*Proof.* By the previous lemma and remark, we see that (a) is equivalent to the following: For any nonzero  $x^* \in F^*$  there exists an  $x \in F$  satisfying  $\langle x, x^* \rangle \neq 0$ , i. e.  $F$  separates  $F^*$ . And this condition is equivalent to (b), as is stated in Introduction. The proof is complete.

**Remark 2.** In the above theorem, the hypothesis that  $F$  is finite dimensional is not omitted. In fact there are many spaces  $E$  such that  $\dim E < \dim E^*$ . If we let  $S = \{I\}$ , where  $I$  denotes the identity operator on such a space  $E$ , then clearly (b) holds but (a) does not.

### 3. Ergodic properties of Markov operator semigroups

Let  $X$  be a compact Hausdorff space and  $C(X)$  the Banach space of all (real or complex) continuous functions on  $X$  with the supremum norm. A linear operator  $T$  on  $C(X)$  is said to be a *Markov operator* if  $T1 = 1$  and if  $f \geq 0$  implies  $Tf \geq 0$ . Let  $\mathfrak{S}$  be a fixed semigroup of Markov operators on  $C(X)$ , and put

$$C_t(X) = \{f \in C(X) : Tf = f \text{ for all } T \in \mathfrak{S}\}.$$

It is well-known ([3], p. 265) that the topological dual space  $C^*(X)$  of  $C(X)$  is identified with the space of all regular finite (countably additive) measures on the  $\sigma$ -field  $\Sigma$  of Borel subsets of  $X$ . Denote by  $\mathcal{P}(X)$  the regular probability measures on  $\Sigma$ , and put

$$\mathcal{P}_i(X) = \{\mu \in \mathcal{P}(X) : T^*\mu = \mu \text{ for all } T^* \in \mathcal{G}^*\}.$$

We define, as in Sine [10], the center  $M$  of  $\mathcal{G}$  by

$$M = \text{closure} \cup \{\text{supp } \mu : \mu \in \mathcal{P}_i(X)\}.$$

A closed subset  $K$  of  $X$  is said to be  $\mathcal{G}$ -invariant if  $\text{supp } T^*e_x \subset K$  for every  $x \in K$ , where  $e_x$  denotes the unit mass concentrated at  $x$ . It is easily seen from Sine [8] that  $M$  is  $\mathcal{G}$ -invariant.

**Proposition.** *Let  $\mathcal{G}$  be a weakly right ergodic semigroup of Markov operators on  $C(X)$ . Then any  $g \in C(X)$  with  $g=0$  on  $M$  is in the closed linear subspace  $N$  of  $C(X)$  determined by the set  $\{f - Tf : f \in C(X) \text{ and } T \in \mathcal{G}\}$ .*

*Proof.* Let  $(T_n, n \in \mathcal{A})$  be a weakly right  $\mathcal{G}$ -ergodic net. If  $\mu \in \mathcal{P}_i(X)$ , then as in the proof of the lemma there exists a subnet  $(T_{n'}, \mu, n' \in \mathcal{A}')$  of the net  $(T_n^* \mu, n \in \mathcal{A})$  and an element  $\tilde{\mu} \in C^*(X)$  such that

$$\text{weak}^*\text{-}\lim_{n'} T_{n'}^* \mu = \tilde{\mu}.$$

It follows that  $T^*\tilde{\mu} = \tilde{\mu}$  for all  $T^* \in \mathcal{G}^*$ . Since  $\|T^*\| = 1$  for all  $T^* \in \mathcal{G}^*$ , it follows that  $\tilde{\mu}$  is a finite linear combination of elements of  $\mathcal{P}_i(X)$ . Hence  $\text{supp } \tilde{\mu} \subset M$ , and so we have

$$\lim_{n'} \langle T_{n'} g, \mu \rangle = \lim_{n'} \langle g, T_{n'}^* \mu \rangle = \langle g, \tilde{\mu} \rangle = 0.$$

By this and an easy induction argument, the zero function 0 is a weak cluster element of the net  $(T_n g, n \in \mathcal{A})$ , and thus we have  $0 \in \overline{A}(g)$ . Therefore given an  $\varepsilon > 0$  there exists an  $h = \sum_{i=1}^k a_i T_i g$  with  $\|h\| < \varepsilon$ , where  $\sum_{i=1}^k a_i = 1$  and  $T_i \in \mathcal{G}$  for each  $i$ . Consequently

$$g = h + \sum_{i=1}^k a_i (g - T_i g),$$

and this proves the proposition.

In Theorem 2 below we study ergodic properties of  $\mathcal{G}$  restricted to the center  $M$ . A semigroup  $\mathcal{G}$  is said to be *continuously scattered* if there exists a family of functions in  $C(X)$  so that each function in the family is constant on the support of each extreme measure of  $\mathcal{P}_i(X)$  and the family separates the extreme measures of  $\mathcal{P}_i(X)$ .

**Theorem 2.** *Let  $\mathcal{G}$  be a semigroup of Markov operators on  $C(X)$  and*

$(T_n, n \in \mathcal{A})$  a right  $\mathfrak{S}$ -ergodic net of linear operators on  $C(X)$ . Then the following conditions are equivalent:

- (a)  $\mathfrak{S}$  is continuously scattered.
- (b) For any  $f \in C(X)$  the net  $(T_n f, n \in \mathcal{A})$  converges uniformly on the center  $M$ , and further  $\lim_n T_n f - T_n f = 0$  uniformly on  $M$  for all  $T \in \mathfrak{S}$ .

*Proof.* We proceed partly as in Sine [10]. Since  $M$  is  $\mathfrak{S}$ -invariant, we may and will assume without loss of generality that  $M$  equals the whole space  $X$ .

(a)  $\Rightarrow$  (b). Let  $\mathcal{A}$  be the family of all  $f \in C(X)$  that are constant on the support of each extreme measure of  $\mathcal{P}_i(X)$ . Then we see that  $\mathcal{A}$  is a norm closed algebra and that if  $f \in \mathcal{A}$  and if  $\mu$  is an extreme measure of  $\mathcal{P}_i(X)$  then  $Tf = f$  on  $\text{supp } \mu$  for all  $T \in \mathfrak{S}$ , because  $\text{supp } \mu$  is  $\mathfrak{S}$ -invariant. By the Krein-Milman theorem, the union

$$\cup \{ \text{supp } \mu : \mu \text{ is an extreme measure of } \mathcal{P}_i(X) \}$$

is dense in  $X (= M)$ , and thus the continuity of  $f$  implies that  $Tf = f$  on  $X$  for all  $T \in \mathfrak{S}$ . Let  $Y$  be the quotient topological space  $X/\mathcal{A}$ . The quotient map  $q$  is defined by

$$q(x) = \{ z \in X : f(z) = f(x) \text{ for all } f \in \mathcal{A} \} \quad (\in Y)$$

for all  $x \in X$ .  $Y$  is then a compact Hausdorff space and  $q$  is continuous. The Stone-Weierstrass theorem implies that  $\mathcal{A}$  can be identified with the Banach space  $C(Y)$ , and from this it may be readily seen that for any  $y \in Y$  the set  $q^{-1}(y)$  is  $\mathfrak{S}$ -invariant. Since by assumption  $\mathfrak{S}$  is continuously scattered, there exists a unique measure  $\mu$  in  $\mathcal{P}_i(X)$  such that  $\text{supp } \mu \subset q^{-1}(y)$ . It follows from Corollary 1 of [7] and the results of the preceding section that  $(T_n f, n \in \mathcal{A})$  converges uniformly on  $q^{-1}(y)$  to a constant function for each  $f \in C(X)$ .

Let

$$F(x) = \lim_n T_n f(x) \quad (x \in X).$$

To prove the uniform convergence of  $(T_n f, n \in \mathcal{A})$  to  $F$ , let  $x \in q^{-1}(y)$ . Since  $(T_n f, n \in \mathcal{A})$  converges uniformly on  $q^{-1}(y)$ , given an  $\varepsilon > 0$  there exists an  $N \in \mathcal{A}$  such that

$$q^{-1}(y) \subset \{ z \in X : |T_N f(z) - F(x)| < \varepsilon \}.$$

The latter set is open, and hence there exists an open set  $U$  in  $Y$  so that

$$q^{-1}(y) \subset q^{-1}(U) \subset \{ z \in X : |T_N f(z) - F(x)| < \varepsilon \}.$$

Since  $T_N f \in \overline{A}(f)$ , we can choose  $\sum_{i=1}^k a_i T_i f \in A(f)$  so that

$$\|T_N f - \sum_{i=1}^k a_i T_i f\| < \varepsilon.$$

It then follows that

$$|\sum_{i=1}^k a_i T_i f - F(x)| < 2\varepsilon \text{ on } q^{-1}(U)$$

and that

$$T_n f = T_n(\sum_{i=1}^k a_i(f - T_i f)) + T_n(\sum_{i=1}^k a_i T_i f - F(x)) + T_n F(x).$$

Since for every  $z \in q^{-1}(U)$  and all  $n \in \mathcal{A}$ ,  $\text{supp } T_n^* e_x \subset q^{-1}(U)$ , we have that

$$\begin{aligned} |T_n f - F(x)| &= |T_n f - T_n F(x)| \\ &\leq \sum_{i=1}^k |a_i| \|T_n f - T_n T_i f\| + 2A\varepsilon \text{ on } q^{-1}(U) \end{aligned}$$

where  $A = \sup_n \|T_n\|$ , and that

$$\lim_n \|T_n f - T_n T_i f\| = 0 \quad (i = 1, \dots, k).$$

Hence we see that  $|F(z) - F(x)| \leq 2A\varepsilon$  for all  $z \in q^{-1}(U)$ , and furthermore that there exists an  $N(x) \in \mathcal{A}$  such that if  $n \geq N(x)$  then

$$|T_n f - F| < 5A\varepsilon \text{ on } q^{-1}(U).$$

Since  $X$  is compact, the uniform convergence of  $(T_n f, n \in \mathcal{A})$  to  $F$  on  $X$  follows. Since  $F \in \mathcal{A} \subset C_i(X)$ , we also have

$$\lim_n \|T T_n f - T_n f\| = \|TF - F\| = 0$$

for all  $T \in \mathfrak{S}$ .

(b)  $\Rightarrow$  (a). If (b) holds then by Corollary 1 of [7]  $C_i(X)$  separates the extreme measures of  $\mathcal{P}_i(X)$ . On the other hand, every  $f \in C_i(X)$  is constant on the support of each extreme measure of  $\mathcal{P}_i(X)$  ([8]). Therefore  $\mathfrak{S}$  is continuously scattered.

The following theorem may be regarded as a generalization of Theorem 3.2 of Atalla [1].

**Theorem 3.** *Let  $\mathfrak{S}$  be a continuously scattered semigroup of Markov operators on  $C(X)$  and  $(T_n, n \in \mathcal{A})$  a right  $\mathfrak{S}$ -ergodic net of linear operators on  $C(X)$ . Then the following conditions are equivalent:*

- (a) *For any  $f \in C(X)$  the net  $(T_n f, n \in \mathcal{A})$  converges uniformly on  $X$ .*
- (b) *There exists a continuous linear operator  $S$  on  $C(X)$  such that for every  $f \in C(X)$   $\lim_n \|S T_n f - T_n f\| = 0$  and such that, for each  $f \in C(X)$  with  $f = 0$  on  $M$ ,  $Sf = 0$  on  $X$ .*

*Proof.* (a) $\Rightarrow$ (b). Let  $Sf = \lim_n T_n f$  for all  $f \in C(X)$ . Since  $(T_n, n \in \mathbb{J})$  is right  $\mathfrak{S}$ -ergodic, if  $f \in C(X)$  satisfies  $f = 0$  on  $M$  then by the Proposition  $Sf = 0$  on  $X$ . Furthermore for every  $T \in \mathfrak{S}$  and all  $f \in C(X)$ ,

$$STf - Sf = \lim_n T_n Tf - T_n f = 0.$$

Hence, immediately,  $ST_n = S$  for all  $n \in \mathbb{J}$ , and so (b) follows.

(b) $\Rightarrow$ (a). Since  $\mathfrak{S}$  is continuously scattered by hypothesis, Theorem 2 shows that the net  $(T_n f, n \in \mathbb{J})$  converges uniformly on the center  $M$  for every  $f \in C(X)$ . Choose an  $F \in C(X)$  so that

$$F(x) = \lim_n T_n f(x) \quad (x \in M).$$

(b) implies that  $\text{supp } S^*e_x \subset M$  for all  $x \in X$ , and hence we have

$$\lim_n \|SF - ST_n f\| \leq \|S\| \lim_n (\sup \{|F(z) - T_n f(z)| : z \in M\}) = 0.$$

Therefore, by (b) again, we have

$$\lim_n \|SF - T_n f\| \leq \lim_n \|SF - ST_n f\| + \lim_n \|ST_n f - T_n f\| = 0,$$

completing the proof.

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